

# BATALIN-VILKOVISKY ALGEBRAS AND CYCLIC COHOMOLOGY OF HOPF ALGEBRAS

LUC MENICHI

**ABSTRACT.** We show that the Connes-Moscovici cyclic cohomology of a Hopf algebra equipped with a character has a Lie bracket of degree  $-2$ . More generally, we show that a "cyclic operad with multiplication" is a cocyclic module whose cohomology is a Batalin-Vilkovisky algebra and whose cyclic cohomology is a graded Lie algebra of degree  $-2$ . This explains why the Hochschild cohomology algebra of a symmetric algebra is a Batalin-Vilkovisky algebra.

## 1. INTRODUCTION

Let  $\mathbb{k}$  be an arbitrary commutative ring and denote by  $\mathcal{H}$  an (ungraded) bialgebra over  $\mathbb{k}$ . We denote by  $\Omega\mathcal{H}$  the Adams Cobar construction on  $\mathcal{H}$ . Its cohomology is  $\text{Cotor}_{\mathcal{H}}^*(\mathbb{k}, \mathbb{k})$ . It results from [10, p. 65] that  $\text{Cotor}_{\mathcal{H}}^*(\mathbb{k}, \mathbb{k})$  has a Gerstenhaber algebra structure.

On the other hand, assume that  $\mathcal{H}$  has an involutive antipode or more generally that  $\mathcal{H}$  is an Hopf algebra equipped with a modular pair in involution where the group like element is the unit 1 of  $\mathcal{H}$ . Connes and Moscovici [3, 4] have proved that  $\Omega\mathcal{H}$  has a canonical cocyclic module structure. Since a cocyclic structure defines a Connes coboundary map  $B$  in cohomology and since a Batalin-Vilkovisky algebra (Definition 3.1) is a Gerstenhaber algebra equipped with an operator  $B$ , it is natural to conjecture that  $\text{Cotor}_{\mathcal{H}}^*(\mathbb{k}, \mathbb{k})$  is a Batalin-Vilkovisky algebra. The first result of this paper is to prove that conjecture.

**Theorem 1.1.** *Let  $\mathcal{H}$  be an Hopf algebra endowed with a modular pair in involution  $(\chi, 1)$ . Then*

*a) The canonical algebra structure of the Cobar construction on  $\mathcal{H}$  together with its Connes-Moscovici cocyclic structure, defines a Batalin-Vilkovisky algebra structure on  $\text{Cotor}_{\mathcal{H}}^*(\mathbb{k}, \mathbb{k})$ .*

*b) The cyclic cohomology of  $\mathcal{H}$ , denoted  $HC_{(\chi, 1)}^*(\mathcal{H})$ , is a graded Lie algebra of degree  $-2$ .*

The easiest way to see that the cotorsion product of a bialgebra  $\mathcal{H}$  is a Gerstenhaber algebra, is to remark as in [10] that the Cobar construction

---

1991 *Mathematics Subject Classification.* 16W30, 19D55, 16E40, 18D50.

*Key words and phrases.* Batalin-Vilkovisky algebra, cyclic operad, cyclic cohomology, Hopf algebra, Hochschild cohomology.

on  $\mathcal{H}$ ,  $\Omega\mathcal{H}$ , is an operad with multiplication (Definition 2.4) and to apply the following general theorem.

- 1.2.** [10, 9, 13] a) *Each operad with multiplication  $O$  is a cosimplicial module. Denote by  $\mathcal{C}^*(O)$  the associated cochain complex.*  
b) *Its cohomology  $H^*(\mathcal{C}^*(O))$  is a Gerstenhaber algebra.*

To prove Theorem 1.1, we proceed similarly:

- we introduce the notion of cyclic operad with multiplication (Definition 3.11),
- we show in section 5 that  $\Omega\mathcal{H}$  is a cyclic operad with multiplication.
- we prove the main result of this paper.

**Theorem 1.3.** *If  $O$  is a cyclic operad with a multiplication then*

- a) *the structure of cosimplicial module on  $O$  extends to a structure of cocyclic module,*
- b) *the Connes coboundary map  $B$  on  $\mathcal{C}^*(O)$  induces a natural structure of Batalin-Vilkovisky algebra on the Gerstenhaber algebra  $H^*(\mathcal{C}^*(O))$  and*
- c) *the cyclic cohomology of  $\mathcal{C}^*(O)$ ,  $HC^*(\mathcal{C}^*(O))$ , has naturally a graded Lie algebra structure of degree  $-2$ .*

Part b) of Theorem 1.3 is inspired by a result (See section 8) announced by McClure and Smith [12].

As a second application of Theorem 1.3, we show

**Theorem 1.4.** *Let  $A$  be an algebra equipped with an isomorphism of  $A$ -bimodules  $\Theta : A \xrightarrow{\cong} A^\vee$  (i. e.  $A$  is a symmetric algebra). Then*

- a) *the Connes coboundary map on  $HH^*(A, A^\vee)$  defines via the isomorphism  $HH^*(A, \Theta) : HH^*(A, A) \xrightarrow{\cong} HH^*(A, A^\vee)$  a structure of Batalin-Vilkovisky algebra on the Gerstenhaber algebra  $HH^*(A, A)$ .*
- b) *The cyclic cohomology of  $A$ ,  $HC^*(A)$  is a graded Lie algebra of degree  $-2$ .*

Part a) of Theorem 1.4 has been proved by Tradler [16]. In fact, he proved part a) much more generally, for "homotopy" symmetric algebras. Our proof for "strict" symmetric algebras is much simpler.

Tamarkin and Tsygan [15, Conjecture 0.13] have conjectured a related result at the chain level. See also McClure and Smith [12]. Moreover Tamarkin and Tsygan have mentioned a relation between Part a) of Theorem 1.4 and Connes-Moscovici cyclic cohomology of Hopf algebras [15]. Theorem 1.3 establishes such relation.

The main tools in the proof of Theorem 1.3 are the following results which have their own interest. Let  $\bar{\mathcal{C}}^*(O)$  be the normalized cochain complex associated to the cyclic operad with multiplication  $O$  and  $B$  the Connes normalized coboundary map on  $\bar{\mathcal{C}}^*(O)$ . Denote by  $\cup$  the cup product and by  $\bar{\circ}$  the composition product in  $\bar{\mathcal{C}}^*(O)$  (See (2.5) and (2.6)).

**Lemma 1.5.** *There is a bilinear map  $Z$  (See (6.1)) of degree  $-1$  such that*

$$B(f \cup g) = Z(f, g) + (-1)^{mn} Z(g, f), \quad \forall f \in \overline{\mathcal{C}}^m(O), g \in \overline{\mathcal{C}}^n(O).$$

**Proposition 1.6.** *There is a bilinear map  $H$  (See (6.4)) of degree  $-2$  such that, for any  $f \in \overline{\mathcal{C}}^m(O)$  and  $g \in \overline{\mathcal{C}}^n(O)$ ,*

$$\begin{aligned} (-1)^m (Z(f, g) - (Bf) \cup g) - f \circ g \\ = dH(f, g) + H(df, g) + (-1)^{m-1} H(f, dg). \end{aligned}$$

We give now the plan of the paper:

**2) operads with multiplication.** This section is a review on operads with multiplication. We recall the definition of operad with multiplication. We define the structure of Gerstenhaber algebra associated to an operad with multiplication. We recall the two fundamental examples of operad with multiplication:

- the endomorphism operad of an algebra,
- the Cobar construction on a bialgebra.

**3) cyclic operad with multiplication.** We introduce the notions of cyclic (non- $\Sigma$ ) operad and cyclic operad with multiplication.

**4) Hochschild cohomology of an symmetric algebra.** We prove Theorem 1.4 by showing that the endomorphism operad of a symmetric algebra is a cyclic operad with multiplication.

**5) Cyclic cohomology of Hopf algebras.** We recall what is an Hopf algebra  $\mathcal{H}$  endowed with a modular pair in involution of the form  $(\chi, 1)$  and we prove Theorem 1.1 by showing that the Cobar construction on  $\mathcal{H}$  is a cyclic operad with multiplication.

**6) Proof of parts a) and b) of Theorem 1.3.** We prove part a) of Theorem 1.3 and Lemma 1.5. Then we deduce part b) from Lemma 1.5 and Proposition 1.6. Finally, we prove Proposition 1.6.

**7) Proof of part c) of Theorem 1.3.** We define the Lie bracket on cyclic cohomology in the same way as Chas and Sullivan define a Lie bracket on  $S^1$ -equivariant homology in [2].

**8) Comparison with McClure and Smith.** We compare Theorem 1.3 with two results announced by McClure and Smith [12].

*Acknowledgment:* We wish to thank Jean-Claude Thomas for his constant support.

## 2. OPERADS WITH MULTIPLICATION

**2.1.** A *Gerstenhaber algebra* is a graded module  $G = \{G^i\}_{i \in \mathbb{Z}}$  equipped with two linear maps

$$\begin{aligned} \cup : G^i \otimes G^j &\rightarrow G^{i+j}, \quad x \otimes y \mapsto x \cup y \\ \{-, -\} : G^i \otimes G^j &\rightarrow G^{i+j-1}, \quad x \otimes y \mapsto \{x, y\} \end{aligned}$$

such that:

- a) the cup product  $\cup$  makes  $G$  into a graded commutative algebra

- b) the bracket  $\{-, -\}$  gives  $G$  a structure of graded Lie algebra of degree  $-1$ . This means that for each  $a, b$  and  $c \in G$   
 $\{a, b\} = -(-1)^{(|a|-1)(|b|-1)}\{b, a\}$  and  
 $\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|-1)(|b|-1)}\{b, \{a, c\}\}.$
- c) the cup product and the Lie bracket satisfy the Poisson rule. This means that for any  $a \in G^k$  the adjunction map  $\{a, -\} : G^i \rightarrow G^{i+k-1}$ ,  $b \mapsto \{a, b\}$  is a  $(k+1)$ -derivation: ie. for  $a, b, c \in G$ ,  $\{a, bc\} = \{a, b\}c + (-1)^{|b|(|a|-1)}b\{a, c\}.$

Usually, this definition is given for a lower graded module  $G = \{G_i\}_{i \in \mathbb{Z}}$ . If you put  $G_i = G^{-i}$  as usual, you pass from an upper degree graded module to a lower graded module and the Lie bracket is of the usual (lower) degree  $+1$ .

**2.2.** In this paper, *operad* means (non- $\Sigma$ ) operad in the category of  $\mathbb{k}$ -modules. That is: a sequence of modules  $\{O(n)\}_{n \in \mathbb{N}}$ , an identity element  $id \in O(1)$  and structure maps

$$\begin{aligned} \gamma : O(n) \otimes O(i_1) \otimes \cdots \otimes O(i_n) &\rightarrow O(i_1 + \cdots + i_n) \\ f \otimes g_1 \otimes \cdots \otimes g_n &\mapsto \gamma(f; g_1, \dots, g_n) \end{aligned}$$

satisfying associativity and unit [11].

Hereafter we use mainly the composition operations  $\circ_i : O(m) \otimes O(n) \rightarrow O(m+n-1)$   $f \otimes g \mapsto f \circ_i g$  defined for  $m \in \mathbb{N}^*$ ,  $n \in \mathbb{N}$  and  $1 \leq i \leq m$  by  $f \circ_i g := \gamma(f; id, \dots, g, id, \dots, id)$  where  $g$  is the  $i$ -th element after the semicolon.

*Example 2.3.* [11] Let  $V$  be a module. The *endomorphism operad* of  $V$  is the operad  $\mathcal{E}nd_V$  defined by  $\mathcal{E}nd_V(n) := \text{Hom}(V^{\otimes n}, V)$ . The identity element of  $\mathcal{E}nd_V$  is the identity map  $id_V : V \rightarrow V$ .

**2.4.** An *operad with multiplication* is a operad equipped with an element  $\mu \in O(2)$  called the multiplication and an element  $e \in O(0)$  such that  $\mu \circ_1 \mu = \mu \circ_2 \mu$  and  $\mu \circ_1 e = id = \mu \circ_2 e$ .

In [10], an operad with multiplication is called a strict unital comp algebra.

Let  $Ass$  be the (non- $\Sigma$ ) associative operad:  $Ass(n) := \mathbb{k}$ . An operad  $O$  is an operad with multiplication if and only if  $O$  is equipped with a morphism of operads  $Ass \rightarrow O$ .

*Sketch of proof of 1.2.* a) The coface maps  $\delta_i : O(n) \rightarrow O(n+1)$  and codegeneracy maps  $\sigma_j : O(n) \rightarrow O(n-1)$  are defined [13] by  $\delta_0 f = \mu \circ_2 f$ ,  $\delta_i f = f \circ_i \mu$ ,  $\delta_{n+1} f = \mu \circ_1 f$  and  $\sigma_{i-1} f = f \circ_i e$  for  $1 \leq i \leq n$ .

b) The associated cochain complex  $\mathcal{C}^*(O)$  is the cochain complex whose differential  $d$  is given by

$$d := \sum_{i=0}^{n+1} (-1)^i \delta_i : O(n) \rightarrow O(n+1).$$

The linear maps  $\cup : O(m) \otimes O(n) \rightarrow O(m+n)$  defined by

$$(2.5) \quad f \cup g := (\mu \circ_1 f) \circ_{m+1} g = (\mu \circ_2 g) \circ_1 f$$

gives  $\mathcal{C}^*(O)$  a structure of differential graded algebra. The linear maps of degree  $-1$

$$\overline{\circ}, \{-, -\} : O(m) \otimes O(n) \rightarrow O(m+n-1)$$

are defined by

$$(2.6) \quad f \overline{\circ} g := (-1)^{(m-1)(n-1)} \sum_{i=1}^m (-1)^{(n-1)(i-1)} f \circ_i g$$

and

$$\{f, g\} := f \overline{\circ} g - (-1)^{(m-1)(n-1)} g \overline{\circ} f.$$

The bracket  $\{-, -\}$  defines a structure of differential graded Lie algebra of degree  $-1$  on  $\mathcal{C}^*(O)$ . After passing to cohomology, the cup product  $\cup$  and the bracket  $\{-, -\}$  satisfy the Poisson rule.  $\square$

**Corollary 2.7.** [8] *The Hochschild cohomology of an algebra is a Gerstenhaber algebra.*

*Proof.* Let  $A$  be an associative algebra with multiplication  $\mu : A \otimes A \rightarrow A$  and unit  $e : \mathbb{k} \rightarrow A$ . Then the endomorphism operad  $\mathcal{E}nd_A$  of  $A$  equipped with  $\mu$  and  $e$  is an operad with multiplication. The Hochschild cochain complex of  $A$ , denoted  $\mathcal{C}^*(A, A)$ , is the cochain complex  $\mathcal{C}^*(\mathcal{E}nd_A)$  associated to the endomorphism operad of  $A$ .  $\square$

**Corollary 2.8.** [10, p. 65] *Let  $\mathcal{H}$  be a bialgebra. Then  $\text{Cotor}_{\mathcal{H}}^*(\mathbb{k}, \mathbb{k})$  is a Gerstenhaber algebra.*

*Proof.* Denote by  $\mu$  and  $1$  the multiplication and the unit of  $\mathcal{H}$ . Denote by  $\Delta$  and  $\varepsilon$  the diagonal and the counit of  $\mathcal{H}$ . For each  $n \in \mathbb{N}$ , denote by  $\Delta^{n-1} : \mathcal{H} \rightarrow \mathcal{H}^{\otimes n}$  the  $(n-1)$  iterated diagonal defined by  $\Delta^{-1} := \varepsilon$ ,  $\Delta^0 := Id_{\mathcal{H}}$  and  $\Delta^{n+1} := (\Delta \otimes id_{\mathcal{H}}^{\otimes n}) \circ \Delta^n$ . For an element  $a \in \mathcal{H}$ , we denote  $\Delta^{n-1} a := a^{(1)} \otimes \cdots \otimes a^{(n)}$  or simply  $a^1 \otimes \cdots \otimes a^n$ . Here the sum is implicit and contrarily to Sweedler notation, we use superscripts instead of subscripts, since we will need indices but no powers.

Consider the operad with multiplication  $O$  defined by  $O(n) := \mathcal{H}^{\otimes n}$  and if  $a_1 \otimes \cdots \otimes a_m \in \mathcal{H}^{\otimes m}$  and  $b_1 \otimes \cdots \otimes b_n \in \mathcal{H}^{\otimes n}$ ,

$$\begin{aligned} (a_1 \otimes \cdots \otimes a_m) \circ_i (b_1 \otimes \cdots \otimes b_n) &:= \\ a_1 \otimes \cdots \otimes a_{i-1} \otimes (\Delta^{n-1} a_i) \cdot (b_1 \otimes \cdots \otimes b_n) \otimes a_{i+1} \otimes \cdots \otimes a_m &= \\ a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i^1 b_1 \otimes \cdots \otimes a_i^n b_n \otimes a_{i+1} \otimes \cdots \otimes a_m. \end{aligned}$$

Here  $\cdot$  is the product obtained by tensorization in  $\mathcal{H}^{\otimes n}$ . The identity element  $id$  of  $O$  is  $1 \in \mathcal{H}^{\otimes 1}$ . The multiplication  $\mu$  is  $1 \otimes 1 \in \mathcal{H}^{\otimes 2}$ . The element  $e$  of  $O$  is the unit of  $\mathbb{k}$ ,  $1_{\mathbb{k}} \in \mathcal{H}^{\otimes 0}$ . The cochain complex associated to this operad is the Cobar construction on  $\mathcal{H}$ , denoted usually  $\Omega\mathcal{H}$ . Since

$\text{Cotor}_{\mathcal{H}}^*(\mathbb{k}, \mathbb{k}) = H^*(\Omega\mathcal{H})$ , the result follows from 1.2. Remark that if  $\mathcal{H}$  is not cocommutative,  $O$  is not in general a symmetric operad.  $\square$

### 3. CYCLIC OPERADS WITH MULTIPLICATION

**3.1.** A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra  $G$  equipped with a degree  $-1$  linear map  $B : G^i \rightarrow G^{i-1}$  such that  $B \circ B = 0$  and

$$(3.2) \quad \{a, b\} = (-1)^{|a|} (B(a \cup b) - (Ba) \cup b - (-1)^{|a|} a \cup (Bb))$$

for  $a$  and  $b \in G$ .

**Definition 3.3.** A *cyclic (non- $\Sigma$ ) operad* is a (non- $\Sigma$ ) operad  $O$  equipped with linear maps  $\tau_n : O(n) \rightarrow O(n)$  for  $n \in \mathbb{N}$  such that

$$(3.4) \quad \forall n \in \mathbb{N}, \quad \tau_n^{n+1} = id_{O(n)},$$

$$(3.5) \quad \forall m \geq 1, n \geq 1, \quad \tau_{m+n-1}(f \circ_1 g) = \tau_n g \circ_n \tau_m f,$$

$$(3.6) \quad \forall m \geq 2, n \geq 0, 2 \leq i \leq m, \quad \tau_{m+n-1}(f \circ_i g) = \tau_m f \circ_{i-1} g,$$

for each  $f \in O(m)$  and  $g \in O(n)$ . In particular, we have  $\tau_1 id = id$ .

This definition is taken from [11, p. 247-8] except that since our operad  $O$  is not necessarily symmetric, we don't assume that the action of the cyclic group  $\mathbb{Z}/(n+1)\mathbb{Z}$  on  $O(n)$  extends to an action of the symmetric group of order  $n+1$ ,  $S_{n+1}$ .

Remark that (3.5) and (3.6) are equivalent to

$$(3.7) \quad \forall m \geq 1, n \geq 1, \quad \tau_{m+n-1}^{-1}(f \circ_m g) = \tau_n^{-1} g \circ_1 \tau_m^{-1} f,$$

$$(3.8) \quad \forall m \geq 2, n \geq 0, 1 \leq i \leq m-1, \quad \tau_{m+n-1}^{-1}(f \circ_i g) = \tau_m^{-1} f \circ_{i+1} g,$$

If instead of (3.5) and (3.6),  $\tau_n$  satisfies

$$\begin{aligned} \tau_{m+n-1}(f \circ_m g) &= \tau_n g \circ_1 \tau_m f, \\ \tau_{m+n-1}(f \circ_i g) &= \tau_m f \circ_{i+1} g, \end{aligned}$$

like in the original definition of cyclic operad of Getzler and Kapranov [5, (2.2)], replace  $\tau_n$  by  $\tau_n^{-1}$ .

We will use the following generalizations of (3.8) and of (3.7): For each  $m \geq 1$ ,  $n \geq 0$ ,  $1 \leq i \leq m$  and  $j \in \mathbb{Z}$ ,

$$(3.9) \quad \text{if } 1 \leq i+j \leq m \text{ then } \tau_{m+n-1}^{-j}(f \circ_i g) = \tau_m^{-j} f \circ_{i+j} g,$$

$$(3.10) \quad \text{if } m+1 \leq i+j \leq m+n \text{ then}$$

$$\tau_{m+n-1}^{-j}(f \circ_i g) = \tau_n^{-j+m-i} g \circ_{i+j-m} \tau_m^{i-m-1} f.$$

**Definition 3.11.** A *cyclic operad with multiplication* is an operad which is both an operad with multiplication and a cyclic operad such that

$$\tau_2 \mu = \mu.$$

The operad  $Ass$  is a cyclic operad: the cyclic group  $\mathbb{Z}/(n+1)\mathbb{Z}$  act trivially on  $Ass(n) := \mathbb{k}$ . A cyclic operad  $O$  is an cyclic operad with multiplication if and only if  $O$  is equipped with a morphism of cyclic operads  $Ass \rightarrow O$ .

#### 4. HOCHSCHILD COHOMOLOGY OF A SYMMETRIC ALGEBRA

**The cyclic endomorphism operad** [11]. Let  $V$  be a module equipped with a bilinear form  $\varphi : V \otimes V \rightarrow \mathbb{k}$  such that the associated right linear map  $\Theta : V \xrightarrow{\cong} V^\vee$ ,  $v \mapsto \varphi(-, v)$ , is an isomorphism. Consider the adjunction map

$$(4.1) \quad Ad : \text{Hom}(V^{\otimes n}, V^\vee) \xrightarrow{\cong} \text{Hom}(V^{\otimes n+1}, \mathbb{k})$$

which associates to any  $g \in \text{Hom}(V^{\otimes n}, V^\vee)$ , the map

$$Ad(g) : V^{\otimes n+1} \rightarrow \mathbb{k}, \quad v_0, v_1, \dots, v_n \mapsto g(v_1, \dots, v_n)(v_0).$$

The composite

$$\text{Hom}(V^{\otimes n}, V) \xrightarrow{\text{Hom}(V^{\otimes n}, \Theta)} \text{Hom}(V^{\otimes n}, V^\vee) \xrightarrow{Ad} \text{Hom}(V^{\otimes n+1}, \mathbb{k})$$

is an isomorphism. Explicitly this composite send  $f \in \text{Hom}(V^{\otimes n}, V)$  to the linear map  $\widehat{f} : V^{\otimes n+1} \rightarrow \mathbb{k}$  defined by

$$\widehat{f}(v_0, v_1, \dots, v_n) = \varphi(v_0, f(v_1, \dots, v_n)) \quad \text{for } v_0, v_1, \dots, v_n \in V.$$

The cyclic group  $\mathbb{Z}/(n+1)\mathbb{Z}$  acts on  $V^{\otimes n+1}$  by permutations of factors:

$$(4.2) \quad t_n(v_0, \dots, v_n) := (v_n, v_0, \dots, v_{n-1}) \quad \text{for } (v_0, \dots, v_n) \in V^{\otimes n+1}.$$

Define  $\tau_n := t_n^\vee : \text{Hom}(V^{\otimes n+1}, \mathbb{k}) \rightarrow \text{Hom}(V^{\otimes n+1}, \mathbb{k})$ . Using the identification  $f \mapsto \widehat{f}$ , we define  $\tau_n : \text{Hom}(V^{\otimes n}, V) \rightarrow \text{Hom}(V^{\otimes n}, V)$  by  $\widehat{\tau_n f} := \tau_n \widehat{f}$  for  $f \in \text{Hom}(V^{\otimes n}, V)$ . Explicitly,  $\tau_n(f)$  is the unique map such that

$$\varphi(v_0, \tau_n f(v_1, \dots, v_n)) = \varphi(v_n, f(v_0, \dots, v_{n-1})) \quad \text{for } v_0, \dots, v_n \in V.$$

The endomorphism operad of  $V$ ,  $\mathcal{E}nd_V$ , equipped with this last linear map  $\tau_n : \mathcal{E}nd_V(n) \rightarrow \mathcal{E}nd_V(n)$  is a cyclic operad if and only if the bilinear form  $\varphi$  is symmetric.

**Hochschild (co)homology.** Let  $A$  be an algebra. Let  $M$  be an  $A$ -bimodule. Denote by  $\mathcal{C}^*(A, M)$  the Hochschild cochain complex of  $A$  with coefficient in  $M$  [7, 1.5.1] and by  $\mathcal{C}_*(A, M)$  the Hochschild chain complex [7, 1.1.1]. Recall that  $\mathcal{C}^n(A, M) := \text{Hom}(A^{\otimes n}, M)$  and that  $\mathcal{C}^n(A, M) := M \otimes A^{\otimes n}$ .

Consider a symmetric algebra  $A$ . By definition, it means that the algebra  $A$  is equipped with an isomorphism  $\Theta : A \xrightarrow{\cong} A^\vee$  of  $A$ -bimodules. By functoriality,  $\mathcal{C}^*(A, \Theta) : \mathcal{C}^*(A, A) \xrightarrow{\cong} \mathcal{C}^*(A, A^\vee)$  is an isomorphism of cosimplicial modules. The adjunction map (4.1)  $Ad : \mathcal{C}^*(A, A^\vee) \xrightarrow{\cong} \mathcal{C}_*(A, A)^\vee$

is an isomorphism of cosimplicial modules (Compare with [7, 1.5.5]). Let  $t_n : \mathcal{C}_n(A, A) \rightarrow \mathcal{C}_n(A, A)$  be the cyclic operator defined by 4.2. The Hochschild chain complex  $\mathcal{C}_*(A, A)$  is a cyclic module [7, 2.1.0]. So  $\mathcal{C}_*(A, A)^\vee$  with  $\tau_n := t_n^\vee$  is a cocyclic module. Therefore by isomorphism,  $\mathcal{C}^*(A, A)$  is also a cocyclic module.

Theorem 1.4 claims that this cocyclic structure on  $\mathcal{C}^*(A, A)$  defines a structure of Batalin-Vilkovisky on the Gerstenhaber algebra  $HH^*(A, A)$ .

*Proof of Theorem 1.4.* Let  $\varphi : A \otimes A \rightarrow \mathbb{k}$  be a bilinear form on  $A$ . It is easy to see that the associated right linear map  $\Theta : A \rightarrow A^\vee$  is a morphism of  $A$ -bimodules if and only if  $\varphi$  is symmetric and

$$(4.3) \quad \varphi(a_2, a_0 a_1) = \varphi(a_0, a_1 a_2), \quad \forall a_0, a_1, a_2 \in A.$$

Therefore the endomorphism operad of the symmetric algebra  $A$ ,  $\mathcal{E}nd_A$  is cyclic: it is the cyclic endomorphism operad defined above. By definition,  $\tau_2 \mu$  is the unique map  $A \otimes A \rightarrow A$  such that

$$\varphi(a_0, \tau_2(\mu)(a_1, a_2)) = \varphi(a_2, a_0 a_1), \quad \forall a_0, a_1, a_2 \in A.$$

Therefore, by (4.3), we have  $\tau_2 \mu = \mu$ . In the proof of Corollary 2.7, we have seen that  $\mathcal{E}nd_A$  is an operad with multiplication. Therefore,  $\mathcal{E}nd_A$  is a cyclic operad with multiplication and by Theorem 1.3,  $HH^*(A, A)$  is a Batalin-Vilkovisky algebra.  $\square$

## 5. CYCLIC COHOMOLOGY OF HOPF ALGEBRAS

Let  $\mathcal{H}$  be an Hopf algebra with antipode  $S$  and unity  $\eta : \mathbb{k} \rightarrow \mathcal{H}$ ,  $\eta(1_{\mathbb{k}}) = 1$ . Consider a morphism of algebras (called *character*)  $\chi : \mathcal{H} \rightarrow \mathbb{k}$ . The *twisted antipode*  $\tilde{S}$  is by definition the convolution product of  $\eta \circ \chi$  and  $S$  in  $\text{Hom}(\mathcal{H}, \mathcal{H})$ . Explicitly, for  $h \in \mathcal{H}$ ,  $\tilde{S}(h) = \chi(h^1)S(h^2)$ , where  $\Delta h = h^1 \otimes h^2$ .

**5.1.** The couple  $(\chi, 1)$  is a *modular pair in involution* for the Hopf algebra  $\mathcal{H}$  if  $\tilde{S} \circ \tilde{S} = id_{\mathcal{H}}$ .

The twisted antipode  $\tilde{S}$  is an algebra antihomomorphism:

$$\tilde{S}(ab) = \tilde{S}(b)\tilde{S}(a), \quad \forall a, b \in \mathcal{H}, \quad \tilde{S}(1) = 1.$$

It is also a coalgebra twisted antihomomorphism:

$$\Delta \tilde{S}(h) = S(h^2) \otimes \tilde{S}(h^1), \quad \forall h \in \mathcal{H}.$$

More generally, we have

$$(5.2) \quad \forall n \geq 1, \quad \Delta \tilde{S}(h) = S(h^n) \otimes \cdots \otimes S(h^2) \otimes \tilde{S}(h^1).$$

Consider the map  $\tau_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}$  defined by

$$\begin{aligned} \tau_n(h_1 \otimes \cdots \otimes h_n) &:= \left( \Delta^{n-1} \tilde{S}(h_1) \right) \cdot (h_2 \otimes \cdots \otimes h_n \otimes 1) \\ &= \tilde{S}(h_1)^1 h_2 \otimes \cdots \otimes \tilde{S}(h_1)^{n-1} h_n \otimes \tilde{S}(h_1)^n. \end{aligned}$$



Here  $\cdot$  is the product in  $\mathcal{H}^{\otimes n}$  and  $\Delta^{n-1}\tilde{S}(h_1) = \tilde{S}(h_1)^1 \otimes \cdots \otimes \tilde{S}(h_1)^n$  (Review the notations introduced in the proof of Corollary 2.8).

In [3, 4], Connes and Moscovici has shown that the Cobar construction on  $\mathcal{H}$  equipped with the maps  $\tau_n$  is a cocyclic module if  $(\delta, 1)$  is a modular pair in involution.

*Proof of Theorem 1.1.* In the proof of Corollary 2.8, we have seen that  $\Omega\mathcal{H}$  is an operad with multiplication. In order to apply Theorem 1.3, we need to see that  $\Omega\mathcal{H}$  is a cyclic operad with multiplication. Therefore it remains to prove (3.5) and (3.6) and that  $\tau_2\mu = \mu$ .

**Proof of (3.5).** Let  $(a_1, \dots, a_m) \in \mathcal{H}^{\otimes m}$  and  $(b_1, \dots, b_n) \in \mathcal{H}^{\otimes n}$ . Since  $\tilde{S}$  is an algebra antihomomorphism and  $\Delta^{m+n-2}$  is an algebra morphism,

$$\Delta^{m+n-2}\tilde{S}(a_1^1 b_1) = \Delta^{m+n-2}\tilde{S}(b_1) \cdot \Delta^{m+n-2}\tilde{S}(a_1^1).$$

$$\begin{aligned} & \text{So } \tau_{m+n-1} [(a_1, \dots, a_m) \circ_1 (b_1, \dots, b_n)] \\ &= \Delta^{m+n-2}\tilde{S}(b_1) \cdot \Delta^{m+n-2}\tilde{S}(a_1^1) \cdot (a_1^2, \dots, a_1^n, 1, \dots, 1) \cdot (b_2, \dots, b_n, a_2, \dots, a_m, 1). \end{aligned}$$

Since  $\Delta$  is coassociative,

$$\left( \tilde{S}(b_1)^{(1)}, \dots, \tilde{S}(b_1)^{(n-1)}, \tilde{S}(b_1)^{(n)1}, \dots, \tilde{S}(b_1)^{(n)m} \right) = \Delta^{m+n-2}\tilde{S}(b_1).$$

$$\begin{aligned} & \text{So } \tau_n(b_1, \dots, b_n) \circ_n \tau_m(a_1, \dots, a_m) \\ &= \Delta^{m+n-2}\tilde{S}(b_1) \cdot (1, \dots, 1, \tilde{S}(a_1)^1, \dots, \tilde{S}(a_1)^m) \cdot (b_2, \dots, b_n, a_2, \dots, a_m, 1). \end{aligned}$$

Therefore to prove (3.5), it suffices to prove that

$$(5.3) \quad \Delta^{m+n-2}\tilde{S}(a_1^1) \cdot (a_1^2, \dots, a_1^n, 1, \dots, 1) = (1, \dots, 1, \tilde{S}(a_1)^1, \dots, \tilde{S}(a_1)^m).$$

Since  $\tilde{S}$  is a twisted antihomomorphism of coalgebras (5.2),

$$\begin{aligned} (5.4) \quad & \Delta^{m+n-2}\tilde{S}(a_1^1) \cdot (a_1^2, \dots, a_1^n, 1, \dots, 1) \\ &= (S(a_1^{(1)m+n-1}) a_1^2, S(a_1^{(1)m+n-2}) a_1^3, \dots, S(a_1^{(1)m+1}) a_1^n, \\ & \quad S(a_1^{(1)m}), \dots, S(a_1^{(1)2}), \tilde{S}(a_1^{(1)1})). \end{aligned}$$

We prove (5.3) by induction on  $n \in \mathbb{N}^*$ :

Case  $n = 1$ . Since  $a_1^1 = \Delta^0 a_1 = a_1$ , the two terms of (5.3) are equal to  $\Delta^{m-1}\tilde{S}(a_1)$ .

Case  $n \geq 2$ . Suppose that (5.3) is true for  $n - 1$ .

$$\Delta^{m+n-2}\tilde{S}(a_1^1) \cdot (a_1^2, \dots, a_1^n, 1, \dots, 1)$$

using (5.4), since  $S$  is an antipode

$$\begin{aligned} &= (\varepsilon(a_1^{(1)m+n-1}) 1, S(a_1^{(1)m+n-2}) a_1^2, \dots, S(a_1^{(1)m+1}) a_1^{n-1}, \\ & \quad S(a_1^{(1)m}), \dots, S(a_1^{(1)2}), \tilde{S}(a_1^{(1)1})) \end{aligned}$$

$$= (1, S\left(\varepsilon\left(a_1^{(1)m+n-1}\right)a_1^{(1)m+n-2}\right)a_1^2, \dots, S\left(a_1^{(1)m+1}\right)a_1^{n-1}, \\ S\left(a_1^{(1)m}\right), \dots, S\left(a_1^{(1)2}\right), \tilde{S}\left(a_1^{(1)1}\right))$$

since  $\varepsilon$  is a counit

$$= (1, S\left(a_1^{(1)m+n-2}\right)a_1^2, \dots, S\left(a_1^{(1)m+1}\right)a_1^{n-1}, \\ S\left(a_1^{(1)m}\right), \dots, S\left(a_1^{(1)2}\right), \tilde{S}\left(a_1^{(1)1}\right))$$

using (5.4) with  $n$  replaced by  $n-1$ ,

$$= (1, \Delta^{m+n-3}\tilde{S}(a_1^1) \cdot (a_1^2, \dots, a_1^{n-1}, 1, \dots, 1))$$

by induction hypothesis

$$= (1, 1, \dots, 1, \tilde{S}(a_1)^1, \dots, \tilde{S}(a_1)^m).$$

**Proof of (3.6).** Since  $\Delta^{n-1}$  is a morphism of algebras,

$$\Delta^{n-1}(\tilde{S}(a_1)^{i-1}a_i) = \Delta^{n-1}(\tilde{S}(a_1)^{i-1}) \cdot \Delta^{n-1}(a_i).$$

$$\text{So } \tau_m(a_1, \dots, a_m) \circ_{i-1} (b_1, \dots, b_n) \\ = (\tilde{S}(a_1)^1 a_2, \dots, \tilde{S}(a_1)^{i-2} a_{i-1}, \\ \Delta^{n-1}(\tilde{S}(a_1)^{i-1}) \cdot \Delta^{n-1}(a_i) \cdot (b_1, \dots, b_n), \\ \tilde{S}(a_1)^i a_{i+1}, \dots, \tilde{S}(a_1)^{m-1} a_m, \tilde{S}(a_1)^m).$$

Since  $\Delta$  is coassociative (case  $n \geq 1$ ) and counitary (case  $n = 0$ ),

$$\tilde{S}(a_1)^1 \otimes \dots \otimes \tilde{S}(a_1)^{i-2} \otimes \Delta^{n-1}\tilde{S}(a_1)^{i-1} \otimes \tilde{S}(a_1)^i \otimes \dots \otimes \tilde{S}(a_1)^m \\ = \Delta^{m+n-2}\tilde{S}(a_1).$$

$$\text{Therefore } \tau_m(a_1, \dots, a_m) \circ_{i-1} (b_1, \dots, b_n) \\ = \Delta^{m+n-2}\tilde{S}(a_1) \cdot (a_2, \dots, a_{i-1}, \Delta^{n-1}(a_i) \cdot (b_1, \dots, b_n), a_{i+1}, \dots, a_m, 1) \\ = \tau_{m+n-1}((a_1, \dots, a_m) \circ_i (b_1, \dots, b_n)).$$

The multiplication  $\mu$  on the operad  $\Omega\mathcal{H}$  is  $1 \otimes 1$ . Since  $\tilde{S}(1) = 1$ , it is easy to check that  $\tau_2\mu = \mu$ .  $\square$

## 6. PROOF OF PARTS A) AND B) OF THEOREM 1.3

*Proof of part a) of Theorem 1.3.* Let  $f \in O(n-1)$ . By (3.5) and  $\tau_2\mu = \mu$

$$\tau_n\delta_1 f = \tau_n(f \circ_1 \mu) = \tau_2\mu \circ_2 \tau_{n-1}f = \delta_0\tau_{n-1}f.$$

By (3.6), for  $2 \leq i \leq n-1$

$$\tau_n\delta_i f = \tau_n(f \circ_i \mu) = \tau_n f \circ_{i-1} \mu = \delta_{i-1}\tau_{n-1}f.$$

By (3.5),

$$\tau_n\delta_n f = \tau_n(\mu \circ_1 f) = \tau_{n-1}f \circ_{n-1} \mu = \delta_{n-1}\tau_{n-1}f.$$

Let  $g \in O(n+1)$ . By (3.6),

$$\tau_n \sigma_j g = \tau_n (g \circ_{j+1} e) = \tau_n g \circ_j e = \sigma_{j-1} \tau_n g.$$

Therefore [7, 6.1.1] the cosimplicial module  $O$  is in fact a cocyclic module.  $\square$

Denote by  $B$  the Connes coboundary map associated to the cocyclic module  $O$ . By 1.2, we already know that  $H(\mathcal{C}^*(O))$  is a Gerstenhaber algebra. Therefore to prove part b) of Theorem 1.3, it suffices to prove that (3.2) holds in cohomology.

**Normalization.** We would like to use the normalized cochain complex instead of the unnormalized one, since the formula for Connes coboundary map  $B$  is simpler in the normalized cochain complex. By definition, the normalized cochain complex associated to  $O$ , denoted  $\overline{\mathcal{C}}^*(O)$ , is the subcomplex of  $\mathcal{C}^*(O)$  defined by

$$\overline{\mathcal{C}}^n(O) := \{f \in \mathcal{C}^n(O) \text{ such that } \sigma_j f = 0 \text{ for } 0 \leq j \leq n-1\}.$$

It is well known that the inclusion  $\overline{\mathcal{C}}^*(O) \xrightarrow{\sim} \mathcal{C}^*(O)$  is a cochain homotopy equivalence. It is easy to see that if  $f \in \overline{\mathcal{C}}^m(O)$  and  $g \in \overline{\mathcal{C}}^n(O)$  then  $f \cup g \in \overline{\mathcal{C}}^{m+n}(O)$  and  $f \circ_i g \in \overline{\mathcal{C}}^{m+n-1}(O)$  for  $1 \leq i \leq m$ . Therefore  $\overline{\mathcal{C}}^*(O)$  is both a subalgebra and a sub Lie algebra of  $\mathcal{C}^*(O)$ . And so, it suffices to show that for any cycles  $f$  and  $g \in \overline{\mathcal{C}}^*(O)$ , (3.2) holds modulo coboundaries.

**Reduction.** In this section, we show that in order to prove (3.2), it suffices to prove Proposition 1.6. The idea behind that reduction is to start by proving the following particular case of (3.2) (where the number of terms has been divided by two): If  $f \in H(\mathcal{C}^*(O))$  is of even degree then  $B(f \cup f)$  is divisible by 2 and

$$f \overline{\circ} f = \frac{1}{2} \{f, f\} = \frac{1}{2} B(f \cup f) - (Bf) \cup f.$$

Proposition 1.6 is a slight generalization of this formula. Lemma 1.5 implies in particular that  $B(f \cup f)$  is a multiple of 2 if  $f$  is of even degree.

The bilinear map of degree  $-1$

$$Z : \overline{\mathcal{C}}^m(O) \otimes \overline{\mathcal{C}}^n(O) \rightarrow \overline{\mathcal{C}}^{m+n-1}(O), \quad f \otimes g \mapsto Z(f, g)$$

is defined by

$$(6.1) \quad Z(f, g) := (-1)^{mn} \sum_{j=1}^m (-1)^{j(m+n-1)} \tau_{m+n-1}^{-j} \sigma_{m+n}(g \cup f).$$

Here  $\sigma_n : O(n) \rightarrow O(n-1)$  is the extradegeneracy operator defined by  $\sigma_n := \sigma_{n-1} \tau_n$ .

In order to prove Lemma 1.5, we need the following two equations

$$(6.2) \quad \sigma_{m+n}(f \cup g) = \tau_m f \circ_m g.$$

*Proof.* By (3.6), (3.5) and since  $\tau_2\mu = \mu$ ,

$$\begin{aligned}\tau_{m+n}(f \cup g) &= \tau_{m+n}((\mu \circ_1 f) \circ_{m+1} g) = \tau_{m+1}(\mu \circ_1 f) \circ_m g \\ &= (\tau_m f \circ_m \tau_2 \mu) \circ_m g = (\tau_m f \circ_m \mu) \circ_m g.\end{aligned}$$

Therefore since  $\mu \circ_2 e = id$

$$\sigma_{m+n}(f \cup g) = [(\tau_m f \circ_m \mu) \circ_m g] \circ_{m+n} e = \tau_m f \circ_m g.$$

□

$$(6.3) \quad \tau_{m+n-1}^{-n} \sigma_{m+n}(f \cup g) = \sigma_{m+n}(g \cup f).$$

*Proof.* Using ((3.10)) and equation (6.2),

$$\tau_{m+n-1}^{-n} \sigma_{m+n}(f \cup g) = \tau_{m+n-1}^{-n} (\tau_m f \circ_m g) = \tau_n^{-n} g \circ_n f = \sigma_{m+n}(g \cup f)$$

□

*Proof of Lemma 1.5.* The operator  $N : O(n-1) \rightarrow O(n-1)$  is defined by

$$N := \sum_{i=0}^{n-1} (-1)^{i(n-1)} \tau_{n-1}^i = \sum_{j=1}^n (-1)^{j(n-1)} \tau_{n-1}^{-j}.$$

By definition, Connes normalized cochain coboundary is  $B := N\sigma_n : O(n) \rightarrow O(n-1)$ . Therefore, using equation (6.3),

$$\begin{aligned}B(f \cup g) &= \sum_{j=1}^n (-1)^{j(m+n-1)} \tau_{m+n-1}^{-j} \sigma_{m+n}(f \cup g) \\ &\quad + \sum_{j=n+1}^{m+n} (-1)^{j(m+n-1)} \tau_{m+n-1}^{-(j-n)} \tau_{m+n-1}^{-n} \sigma_{m+n}(f \cup g) \\ &= \sum_{j=1}^n (-1)^{j(m+n-1)} \tau_{m+n-1}^{-j} \sigma_{m+n}(f \cup g) \\ &\quad + \sum_{j=1}^m (-1)^{(j+n)(m+n-1)} \tau_{m+n-1}^{-j} \sigma_{m+n}(g \cup f) \\ &= (-1)^{mn} Z(g, f) + Z(f, g)\end{aligned}$$

□

**6.4.** Let  $f \in \overline{\mathcal{C}}^m(O)$  and  $g \in \overline{\mathcal{C}}^n(O)$ . Define for any  $1 \leq j \leq p \leq m-1$

$$H_{j,p}(f, g) := (-1)^{j^{m-j+(n-1)(p+1+m)}} \tau_{m+n-2}^{-j} \sigma_{m+n-1}(f \circ_{p-j+1} g),$$

and consider the bilinear map of degree  $-2$ ,

$$\begin{aligned}H : \overline{\mathcal{C}}^m(O) \otimes \overline{\mathcal{C}}^n(O) &\rightarrow \overline{\mathcal{C}}^{m+n-2}(O), \\ f \otimes g &\mapsto H(f, g) := \sum_{1 \leq j \leq p \leq m-1} H_{j,p}(f, g).\end{aligned}$$

*Proof of part b) of Theorem 1.3 assuming Proposition 1.6.* By applying Proposition 1.6 and Lemma 1.5,

$$\begin{aligned}
 & f\overline{\circ}g + \varepsilon g\overline{\circ}f + dH(f, g) + \varepsilon dH(g, f) \\
 & + H(df, g) + \varepsilon H(dg, f) + (-1)^{m-1}H(f, dg) + \varepsilon(-1)^{n-1}H(g, df) \\
 & = (-1)^m Z(f, g) + \varepsilon(-1)^n Z(g, f) - (-1)^m(Bf) \cup g - \varepsilon(-1)^n(Bg) \cup f \\
 & = (-1)^m \left( B(f \cup g) - (Bf) \cup g - (-1)^{m(n-1)}(Bg) \cup f \right).
 \end{aligned}$$

Here the sign  $\varepsilon$  is equal to  $-(-1)^{(m-1)(n-1)} = (-1)^{mn+m+n}$ . Since in cohomology, the cup product is graded commutative, relation (3.2) is proved.  $\square$

**Proof of Proposition 1.6.** Recall that since  $f \in O(m)$ ,

$$df = \mu \circ_2 f + \sum_{i=1}^m (-1)^i f \circ_i \mu + (-1)^{m+1} \mu \circ_1 f.$$

It is easy to see that Proposition 1.6 is a consequence of the following six equations.

$$(6.5) \quad (-1)^m Z(f, g) - f\overline{\circ}g = H(\mu \circ_2 f, g) + H((-1)^{m+1} \mu \circ_1 f, g).$$

$$(6.6) \quad \sum_{1 \leq j < p \leq m} H_{j,p}((-1)^{p-j} f \circ_{p-j} \mu, g) = (-1)^m H(f, \mu \circ_2 g).$$

$$(6.7) \quad \sum_{1 \leq j \leq p \leq m-1} H_{j,p}((-1)^{p-j+1} f \circ_{p-j+1} \mu, g) = (-1)^m H(f, (-1)^{n+1} \mu \circ_1 g).$$

$$(6.8) \quad \sum_{1 \leq j \leq m} H_{j,m}((-1)^{m-j+1} f \circ_{m-j+1} \mu, g) = -(-1)^m (Bf) \cup g.$$

$$\begin{aligned}
 (6.9) \quad & \sum_{1 \leq j \leq p \leq m} H_{j,p} \left( \sum_{\substack{1 \leq i \leq m, \\ i \neq p-j, i \neq p-j+1}} (-1)^i f \circ_i \mu, g \right) \\
 & = -\mu \circ_2 H(f, g) - (-1)^{m+n-1} \mu \circ_1 H(f, g) \\
 & \quad - \sum_{1 \leq j \leq p \leq m-1} \sum_{\substack{1 \leq i \leq p-1 \text{ or} \\ p+n \leq i \leq m+n-2}} (-1)^i H_{j,p}(f, g) \circ_i \mu.
 \end{aligned}$$

$$(6.10) \quad \sum_{\substack{1 \leq j \leq p \leq m-1 \\ p \leq i \leq p+n-1}} (-1)^i H_{j,p}(f, g) \circ_i \mu = (-1)^m H \left( f, \sum_{i=1}^n (-1)^i g \circ_i \mu \right).$$

*Proof of (6.5).* By separating the terms  $j = p$  and  $j < p$ ,

$$\begin{aligned} H(\mu \circ_2 f, g) &= \sum_{1 \leq j \leq p \leq m} (-1)^{jm+(n-1)(p+m)} \tau_{m+n-1}^{-j} \sigma_{m+n}((\mu \circ_2 f) \circ_{p-j+1} g) \\ &= (-1)^m Z(f, g) \\ &\quad + \sum_{1 \leq j < p \leq m} (-1)^{jm+(n-1)(p+m)} \tau_{m+n-1}^{-j} \sigma_{m+n}(id \cup f \circ_{p-j} g). \end{aligned}$$

On the other hand, since (6.3)  $\tau_{m+n-1}^{-1} \sigma_{m+n}(f \circ_{p-j+1} g \cup id) = \sigma_{m+n}(id \cup f \circ_{p-j+1} g)$ ,

$$\begin{aligned} &(-1)^{m+1} H(\mu \circ_1 f, g) \\ &= \sum_{1 \leq j \leq p \leq m} (-1)^{m+1+jm+(n-1)(p+m)} \tau_{m+n-1}^{-(j-1)} \sigma_{m+n}(id \cup f \circ_{p-j+1} g). \end{aligned}$$

Therefore, since (6.2)  $\sigma_{m+n}(id \cup f \circ_p g) = \tau_1 id \circ_1 (f \circ_p g) = f \circ_p g$ , by the change of variables  $j' = j - 1$ ,

$$\begin{aligned} &(-1)^{m+1} H(\mu \circ_1 f, g) = -f \circ g \\ &\quad - \sum_{1 \leq j' < p \leq m} (-1)^{j'm+(n-1)(p+m)} \tau_{m+n-1}^{-j'} \sigma_{m+n}(id \cup f \circ_{p-j'} g). \end{aligned}$$

□

*Proof of (6.6).* By the change of variables  $p' = p - 1$ ,

$$\begin{aligned} &\sum_{1 \leq j < p \leq m} H_{j,p}((-1)^{p-j} f \circ_{p-j} \mu, g) \\ &= \sum_{1 \leq j \leq p' \leq m-1} (-1)^{p'+1-j+jm+(n-1)(p'+1+m)} \tau_{m+n-1}^{-j} \sigma_{m+n}(f \circ_{p'+1-j} (\mu \circ_2 g)) \\ &= (-1)^m H(f, \mu \circ_2 g). \end{aligned}$$

□

The proof of (6.7) is similar.

To prove the last three equations, we will express all the formulas in terms of composite  $\circ_i$  of the elements  $\tau_m^{-j} f$ ,  $g$ ,  $\mu$  and  $e$ , using again and again (3.10) and (3.9). Therefore, we start by giving a new expression for  $H_{j,p}(f, g)$ :

$$(6.11) \quad H_{j,p}(f, g) = (-1)^{jm-j+(n-1)(p+1+m)} \sigma_{j-1}(\tau_m^{-j} f \circ_{p+1} g).$$

*Proof.* We have seen that  $O$  is a cocyclic module. Therefore [1, Remark 1.2], the following relation between  $\tau_n$  and the degeneracy maps  $\sigma_i$  holds

$$\forall 0 \leq r \leq i \leq n, \quad \tau_n^r \sigma_i = \sigma_{i-r} \tau_{n+1}^r.$$

For the extra degeneracy map  $\sigma_{n+1}$ , we have

$$\forall 0 \leq r \leq n, \quad \tau_n^r \sigma_{n+1} = \sigma_{n-r} \tau_{n+1}^{r+1}$$

or equivalently

$$(6.12) \quad \forall 1 \leq j \leq n+1, \quad \tau_n^{-j} \sigma_{n+1} = \sigma_{j-1} \tau_{n+1}^{-j}.$$

Therefore using (3.9),

$$\tau_{m+n-2}^{-j} \sigma_{m+n-1} (f \circ_{p-j+1} g) = \sigma_{j-1} \tau_{m+n-1}^{-j} (f \circ_{p-j+1} g) = \sigma_{j-1} (\tau^{-j} f \circ_{p+1} g).$$

□

*Proof of (6.8).* By (6.12),

$$B(f) = \sum_{j=1}^m (-1)^{j(m-1)} \sigma_{j-1} \tau_m^{-j} f.$$

By (6.11) and (3.10),

$$\begin{aligned} & \sum_{1 \leq j \leq m} H_{j,m} ((-1)^{m-j+1} f \circ_{m-j+1} \mu, g) \\ &= \sum_{j=1}^m (-1)^{m+1+jm+j} \sigma_{j-1} [(\tau_2^{-1} \mu \circ_1 \tau_m^{-j} f) \circ_{m+1} g] \\ &= -(-1)^m (Bf) \cup g. \end{aligned}$$

□

*Proof of (6.9).* In all this proof, we put  $\varepsilon := (-1)^{i+jm+(n-1)(p+m)}$ . By (6.11),

$$\begin{aligned} & \sum_{1 \leq j \leq p \leq m} H_{j,p} \left( \sum_{\substack{1 \leq i \leq m, \\ i \neq p-j, i \neq p-j+1}} (-1)^i f \circ_i \mu, g \right) \\ &= \sum_{\substack{1 \leq j \leq p \leq m, 1 \leq i \leq m, \\ i+j < p \text{ or } i+j > p+1}} \varepsilon \sigma_{j-1} \left[ \tau_{m+1}^{-j} (f \circ_i \mu) \circ_{p+1} g \right]. \end{aligned}$$

Remark that when  $m+1 \leq i+j$ , we can forget the condition  $i+j \neq p$  under theses sums, and that when  $m+2 \leq i+j$ , we can also forget the condition  $i+j \neq p+1$ .

Using respectively (3.9), (3.10), (3.10) again and (3.10) twice, we obtain that

$$\tau_{m+1}^{-j} (f \circ_i \mu) = \begin{cases} \tau_m^{-j} f \circ_{i+j} \mu & \text{if } i+j \leq m, \\ \mu \circ_1 \tau_m^{-j} f & \text{if } i+j = m+1, \\ \mu \circ_2 \tau_m^{-(j-1)} f & \text{if } i+j = m+2, \\ \tau_m^{-(j-1)} f \circ_{i+j-m-2} \mu & \text{if } m+3 \leq i+j. \end{cases}$$

By the change of variables  $i' = i + j$ , we have  $\varepsilon = (-1)^{i'-j+jm+(n-1)(p+m)}$ ,

$$\begin{aligned}
& \sum_{\substack{1 \leq j \leq p \leq m, \\ 1 \leq i < p-j}} \varepsilon \sigma_{j-1} [(\tau_m^{-j} f \circ_{i+j} \mu) \circ_{p+1} g] \\
&= \sum_{1 \leq j < i' < p \leq m} \varepsilon \gamma(\tau_m^{-j} f; id, \dots, id, e, id, \dots, id, \mu, id, \dots, id, g, id, \dots, id) \\
&= - \sum_{1 \leq j \leq i < p \leq m-1} (-1)^i H_{j,p}(f, g) \circ_i \mu,
\end{aligned}$$

where in the second sum,  $e$  is the  $j$ -th element after the semi-colon,  $\mu$  is the  $i'$ -th element and  $g$  is the  $p$ -th element. And we have

$$\begin{aligned}
& \sum_{\substack{1 \leq j \leq p \leq m, \\ p-j+1 < i \leq m-j}} \varepsilon \sigma_{j-1} [(\tau_m^{-j} f \circ_{i+j} \mu) \circ_{p+1} g] \\
&= \sum_{\substack{1 \leq j \leq p \leq m, \\ p+1 < i' \leq m}} \varepsilon \gamma(\tau_m^{-j} f; id, \dots, id, e, id, \dots, id, g, id, \dots, id, \mu, id, \dots, id) \\
&= - \sum_{\substack{1 \leq j \leq p \leq m-1, \\ p+n \leq i \leq m+n-2}} (-1)^i H_{j,p}(f, g) \circ_i \mu,
\end{aligned}$$

where in the second sum,  $e$  is the  $j$ -th element after the semi-colon,  $g$  is the  $(p+1)$ -th element and  $\mu$  is the  $i'$ -th element.

For  $k = 1$  or  $k = 2$ ,

$$\mu \circ_k H(f, g) = \sum_{1 \leq j \leq p \leq m-1} (-1)^{jm-j+(n-1)(p+1+m)} \mu \circ_k [(\tau_m^{-j} f \circ_{p+1} g) \circ_j e].$$

Since when  $1 \leq j \leq p \leq m$  and  $i + j = m + 1$ , we have  $1 \leq i \leq m$  and the equivalence

$$i + j \neq p + 1 \iff p \neq m,$$

$$\sum_{\substack{1 \leq j \leq p \leq m, 1 \leq i \leq m, \\ i+j=m+1, i+j \neq p+1}} \varepsilon \sigma_{j-1} [(\mu \circ_1 \tau_m^{-j} f) \circ_{p+1} g] = -(-1)^{m+n-1} \mu \circ_1 H(f, g).$$

Since when  $1 \leq j \leq p \leq m$  and  $i + j = m + 2$ , we have the equivalence

$$1 \leq i \leq m \iff 2 \leq j,$$

by the change of variables  $j' = j - 1$  and  $p' = p - 1$ ,

$$\sum_{\substack{1 \leq j \leq p \leq m, 1 \leq i \leq m, \\ i+j=m+2}} \varepsilon \sigma_{j-1} [(\mu \circ_2 \tau_m^{-(j-1)} f) \circ_{p+1} g] = -\mu \circ_2 H(f, g).$$



By the change of variables  $j' = j - 1$ ,  $i' = i + j - m - 2$  and then  $p' = p - 1$ , we have  $\varepsilon = (-1)^{i'-j'-1+j'm+(n-1)(p+m)}$  and

$$\begin{aligned} & \sum_{\substack{1 \leq j \leq p \leq m, \\ m+3-j \leq i \leq m}} \varepsilon \sigma_{j-1} \left[ (\tau_m^{-(j-1)} f \circ_{i+j-m-2} \mu) \circ_{p+1} g \right] \\ &= \sum_{1 \leq i' < j' < p \leq m} \varepsilon \gamma(\tau_m^{-j} f; id, \dots, id, \mu, id, \dots, id, e, id, \dots, id, g, id, \dots, id) \\ &= - \sum_{1 \leq i' < j' \leq p' \leq m-1} (-1)^{i'} H_{j', p'}(f, g) \circ_{i'} \mu, \end{aligned}$$

where in the second sum,  $\mu$  is the  $i'$ -th element after the semi-colon,  $e$  is the  $j'$ -th element and  $g$  the  $p$ -th element.  $\square$

To prove (6.10), use (6.11) and the change of variables  $i' = i - p + 1$ .

## 7. PROOF OF PART C) OF THEOREM 1.3

Using the following Proposition, we see immediately that part c) of Theorem 1.3 follows from parts a) and b).

In this section, all the graded modules are considered as lower graded. Recall that a *mixed complex* is a graded module  $M = \{M_i\}_{i \in \mathbb{Z}}$  equipped with a linear map of degree  $-1$   $d : M_i \rightarrow M_{i-1}$  and a linear map of degree  $+1$   $B : M_i \rightarrow M_{i+1}$  such that  $d^2 = B^2 = dB + Bd = 0$ .

**Proposition 7.1.** *Let  $(M, d, B)$  be a mixed complex such that its homology  $H_*(M, d)$  equipped with  $H_*(B)$  has a Batalin-Vilkovisky algebra structure. Then its cyclic homology  $HC_*(M)$  is a graded Lie algebra of lower degree  $+2$ .*

The key point in the proof of this proposition is the following lemma not explicitated stated in [2]. The proof of this lemma is exactly the proof of Theorem 6.1 of [2].

**Lemma 7.2.** *Let  $H$  be a Batalin-Vilkovisky algebra and  $HC$  be a graded module. Consider a long exact sequence of the form*

$$\cdots \rightarrow H_n \xrightarrow{I} HC_n \rightarrow HC_{n-2} \xrightarrow{\partial} H_{n-1} \rightarrow \cdots$$

*If the operator  $B : H_i \rightarrow H_{i-1}$  is equal to  $\partial \circ I$  then*

$$[a, b] := (-1)^{|a|} I(\partial a \cup \partial b), \quad \forall a, b \in HC$$

*defines a Lie bracket of degree  $+2$  on  $HC$ .*

*Proof of Proposition 7.1.* A mixed complex  $M$  is a (differential graded) module over the differential exterior graded algebra  $\Lambda := (\Lambda_{\varepsilon_1}, 0)$  [6]. Consider the Bar construction of  $\Lambda$  with coefficients in  $M$ ,  $B(M; \Lambda; \mathbb{k})$ . By [6, Proposition 1.4], the cyclic homology of  $M$ , is the homology of  $B(M; \Lambda; \mathbb{k})$ :

$$HC_*(M) := H_*(B(M; \Lambda; \mathbb{k})) = \text{Tor}_*^\Lambda(M, \mathbb{k}).$$

Explicitly,  $B(M; \Lambda; \mathbb{k})$  is the complex defined as follow:

$$B(M; \Lambda; \mathbb{k})_n = M_n \oplus M_{n-2} \oplus M_{n-4} \oplus \cdots \quad \text{and} \\ d(m_n, m_{n-2}, m_{n-4}, \cdots) = (dm_n + Bm_{n-2}, dm_{n-2} + Bm_{n-4}, \cdots).$$

A mixed complex yields a Connes long exact sequence

$$\cdots \rightarrow H_n(M, d) \xrightarrow{I} HC_n(M) \xrightarrow{S} HC_{n-2}(M) \xrightarrow{\partial} H_{n-1}(M, d) \rightarrow \cdots$$

(Usually  $\partial$  is unfortunately denoted by  $B$ , since it is induced by  $B$ .) The connecting homomorphism  $\partial : H_{n-2}(B(M, \Lambda, \mathbb{k})) \rightarrow H_{n-1}(M, d)$  maps the class of the  $n-2$  cycle  $(m_{n-2}, m_{n-4}, \cdots)$  to the class of the cycle  $Bm_{n-2}$  [14, Proof of Prop 2.3.6]. Of course,  $I : H_n(M, d) \rightarrow H_n(B(M, \Lambda, \mathbb{k}))$  maps the class of the cycle  $m_n$  to  $(m_n, 0, 0, \cdots)$ . Therefore  $H_*(B) = \partial \circ I$  [17, Ex. 9.8.2]. Finally by Lemma 7.2,  $HC_*(M)$  is a graded Lie algebra of degree  $+2$ .  $\square$

## 8. COMPARISON WITH McCLURE AND SMITH

In this section, we compare Part b) of Theorem 1.3 with a result announced by McClure and Smith [12]:

A cosimplicial module (resp. space)  $X^\bullet$  has a *cup-cocyclic* structure if it is a cocyclic module (resp. space) and if it has a cup product [13, Definition 2.1(iii)] such that

$$\tau_{m+n+1}^{-n-1}(f \cup \delta_0 g) = g \cup \delta_0 f, \quad \forall f \in X^m, g \in X^n.$$

**8.1.** [12, Corollary] *If the cosimplicial module  $X^\bullet$  has a cup-cocyclic structure then the normalized cochain complex associated,  $\overline{\mathcal{C}}^*(X^\bullet)$ , has an action by an operad equivalent to the singular chains on the operad  $\mathcal{D}$  of framed little disks.*

So  $H^*(\overline{\mathcal{C}}^*(X^\bullet))$  has an action by the operad  $H_*(\mathcal{D})$ , i. e.  $H^*(\overline{\mathcal{C}}^*(X^\bullet))$  is a Batalin-Vilkovisky algebra. Tedious computations show that a cosimplicial module (resp. space) has a cup-cocyclic structure if and only if it is a linear (resp. topological) cyclic operad with multiplication in our sense. Therefore their result gives a Deligne's version of our Theorem.

Note that McClure and Smith have announced a topological counterpart to their result:

**8.2.** [12, Corollary] *If the cosimplicial space  $X^\bullet$  has a cup-cocyclic structure then its realisation,  $Tot(X^\bullet)$ , has an action by an operad equivalent to the operad  $\mathcal{D}$  of framed little disks.*

## REFERENCES

1. D. Burghelea and Z. Fiedorowicz, *Cyclic homology and algebraic K-theory of spaces. II*, Topology **25** (1986), no. 3, 303–317.
2. M. Chas and D. Sullivan, *String topology*, preprint: math.GT/991159, 1999.
3. A. Connes and H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*, Comm. Math. Phys. **198** (1998), no. 1, 199–246.

4. ———, *Cyclic cohomology and Hopf algebra symmetry*, Conférence Moshé Flato 1999, Vol. I (Dijon), Math. Phys. Stud., vol. 21, Kluwer, 2000, pp. 199–246.
5. Getzler E. and Kapranov M., *Cyclic operads and cyclic homology*, Conf. Proc. Lecture Notes Geom. Topology, IV, Internat. Press, 1995, pp. 167–201.
6. C. Kassel, *Cyclic homology, comodules, and mixed complexes*, J. Algebra **107** (1987), no. 1, 195–216.
7. J. Loday, *Cyclic homology*, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 301, Springer-Verlag, Berlin, 1998.
8. Gerstenhaber M., *The cohomology structure of an associative ring*, Ann. of Math. **78** (1963), no. 2, 267–288.
9. Gerstenhaber M. and Voronov A., *Homotopy G-algebras and moduli space operad*, Internat. Math. Res. Notices (1995), no. 3, 141–153.
10. Gerstenhaber M. and Schack S., *Algebras, bialgebras, quantum groups, and algebraic deformation*, Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), Contemp. Math., vol. 134, Amer. Math. Soc., 1992, pp. 51–92.
11. Markl M., Shnider S., and Stasheff J., *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, Amer. Math. Soc., 2002.
12. J. McClure and J. Smith, *Batalin-Vilkovisky structures in Hochschild cohomology*, Slides from Northwestern University Algebraic Topology Conference March 24–28, 2002, <http://www.math.northwestern.edu/~pgoerss/emphasis/schedule.html>, 2002.
13. ———, *A solution of Deligne’s hochschild cohomology conjecture*, Contemp. Math., vol. 293, pp. 153–193, Amer. Math. Soc., 2002.
14. P. Seibt, *Cyclic homology of algebras*, World Scientific Publishing Co., 1987.
15. D. Tamarkin and B. Tsygan, *Noncommutative differential calculus, homotopy BV algebras and formality conjectures*, Methods Funct. Anal. Topology **6** (2000), no. 2, 85–100.
16. T. Tradler, *The BV algebra on Hochschild cohomology induced by infinity inner products*, preprint: math.QA/0210150, 2002.
17. C. Weibel, *An introduction to homological algebra*, Cambridge University Press, 1994.

UMR 6093 ASSOCIÉE AU CNRS, UNIVERSITÉ D’ANGERS, FACULTÉ DES SCIENCES, 2  
BOULEVARD LAVOISIER, 49045 ANGERS, FRANCE  
E-mail address: Luc.Menichi@univ-angers.fr